§1 Over $\mathbb{C}$.

Definition

Define $Y(1) = \{$ isom class of elliptic curve $/\mathbb{C}\}$.

as set.

Write $E \cong E_\tau = \mathbb{C}/\underbrace{\mathbb{Z}+\tau\mathbb{Z}}_{\substack{\shortparallel \\ \Lambda_\tau}}$. $\tau \in \mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

Then any gp hom is of the form

$$\begin{aligned} E_{\tau_1} &\xrightarrow{f_\alpha} E_{\tau_2} \\ z &\longmapsto \alpha z. \end{aligned}$$

where $\alpha \Lambda_{\tau_1} \subseteq \Lambda_{\tau_2}$.

Write

$$\left.\begin{aligned} \alpha\tau_1 &= a\tau_2 + b \\ \alpha &= c\tau_2 + d \end{aligned}\right\} \Longrightarrow \tau_1 = \frac{a\tau_2+b}{c\tau_2+d}.$$

f_α is an isom $\iff \alpha\Lambda_{\tau_1} = \Lambda_{\tau_2}$

$\iff \left|\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right| = 1$.

$\left(\Rightarrow \det\begin{pmatrix} a & b \\ c & d\end{pmatrix} = 1 \text{ by } \tau_1, \tau_2 \in \mathfrak{h}\right)$.

Define linear fraction transform.

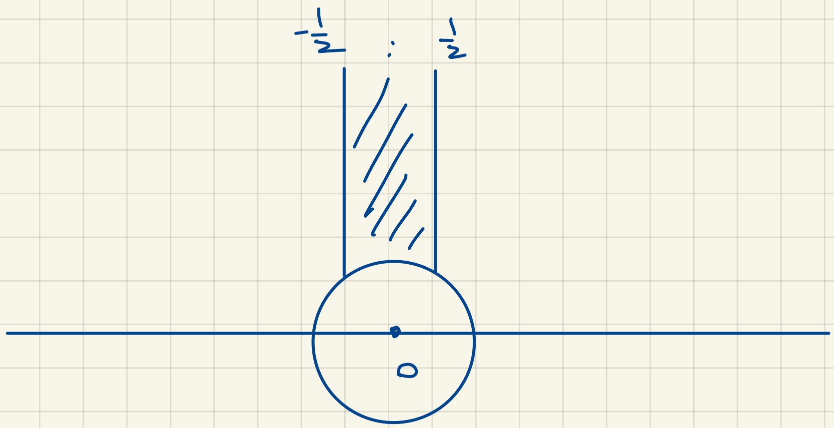
For $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$, $\tau \in \mathfrak{h}$

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\tau = \frac{a\tau+b}{c\tau+d} \in \mathfrak{h}.$$

Then we identify $Y(1) = \mathfrak{h}/SL_2(\mathbb{Z})$.

gives topology.

A fundamental domain is



- $\Gamma(N)$, $\Gamma_1(N)$, $\Gamma_0(N)$:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \bmod N \right\} \quad \left(\Gamma(1) \triangleq SL_2(\mathbb{Z}) \right)$$
$$\cap$$
$$\Gamma_1(N) = \left\{ \qquad \equiv \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \qquad \right\}$$
$$\cap$$
$$\Gamma_0(N) = \left\{ \qquad \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \qquad \right\}$$

Then one can check

$$\mathfrak{h}/\Gamma(N) \longrightarrow \left\{ (E, P, Q) \,\middle|\, \{P, Q\} \text{ is a basis of } E[N] \right\}$$
$$\tau \longmapsto \left(E_\tau, \tfrac{1}{N}, \tfrac{\tau}{N}\right).$$

$$\mathfrak{h}/\Gamma_1(N) \longrightarrow \left\{ (E, P) \,\middle|\, P \text{ is of order } N \right\}$$
$$\tau \longmapsto \left(E_\tau, \tfrac{1}{N}\right)$$

$$\mathfrak{h}/\Gamma_0(N) \longrightarrow \left\{ (E, G) \,\middle|\, G \leq E \text{ cyclic of order } N \right\}.$$
$$\tau \longmapsto \left(E_\tau, \langle \tfrac{1}{N} \rangle\right)$$

Compactification

From the fundamental domain, $Y(1)$ is not cpt.
It becomes cpt after we add $\{\infty\}$

Thus consider $\mathfrak{h}^* = \mathfrak{h} \cup SL_2(\mathbb{Z}) \cdot \{\infty\}$ $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c} \right)$

$$= \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$$

topology : translate $\{ z \mid \operatorname{Im}(z) > K \}$ to each pt.

For $\Gamma = \Gamma(N)$ or $\Gamma_1(N)$ or $\Gamma_0(N)$.
Define

$$X_\Gamma = \mathfrak{h}^* / SL_2(\mathbb{Z}). \quad \text{cpt. Riemann surface.}$$

Note from picture, $g(X_{\Gamma(1)}) = 0$.

$$\begin{array}{c} X_{\Gamma(1)} \\ \triangleq \\ X(1) \end{array}.$$

Stabilizer.

To get genus of other X_p, we shall study the ramification index $X_p \longrightarrow X(1)$. The information will be derived from the stabilizer.

Recall.

$$\begin{array}{ccc} \mathrm{Aut}(E_\tau) & & \\ \text{\rotatebox{90}{$\cong$}} & & \\ \{\alpha \mid \alpha\Lambda_\tau = \Lambda_\tau\} & \xleftrightarrow{\text{bij}} & \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \tau \right\}. \\ \alpha & \longmapsto & \alpha\tau = a\tau + b \\ & & \alpha = c\tau + d. \end{array}$$

- If $c = 0$, $\alpha = \pm 1$.
- If $c \neq 0$, $\tau(c\tau + d) = a\tau + b \Rightarrow [\mathbb{Q}(\tau) : \mathbb{Q}] = 2$.

$$[\mathbb{Q}(\tau):\mathbb{Q}] = [\mathbb{Q}(\alpha) : \mathbb{Q}]$$

$\mathbb{Q}(\tau)$ is quadratic imaginary ext $(\tau \notin \mathbb{R})$.

α is integral sine $\mathbb{Z}[\alpha] \subseteq \Lambda_\tau$ is f.g. $/\mathbb{Z}$.

So is $\alpha^{-1} \Longrightarrow \alpha \in \mathcal{O}^*_{\mathbb{Q}(\tau)}$.

Dirichlet unit thm $\Rightarrow \mathcal{O}^*_{\mathbb{Q}(\tau)} \subseteq$ roots of unity.

$\Rightarrow \alpha = \pm i, \pm\rho = e^{2\pi i/3}$.

So $\mathrm{Aut}(E_\tau) \supsetneq \{\pm 1\} \Longrightarrow \mathrm{End}(E_\tau) = \underbrace{\mathbb{Z}[i] \text{ or } \mathbb{Z}[\rho]}_{= \mathcal{O}_{\mathbb{Q}(\tau)}}$.

Prop

$\mathrm{End}(E) \cong \mathbb{Z}[i] \implies E \cong E_i$

$\mathrm{End}(E) \cong \mathbb{Z}[\rho] \implies E \cong E_\rho.$

pf

$\mathbb{Z}[i]$, $\mathbb{Z}[\rho]$ are PID. Denote $\beta = i$ or ρ.

Say $E \cong E_\tau$, then Λ_τ is a f.g. module / PID.

$\mathrm{rank}_{\mathbb{Z}} \Lambda_\tau = 2$ and Λ_τ torsion free

$\implies \Lambda_\tau$ is free of rank 1 over $\mathbb{Z}[\rho]$

$\implies \exists\, c \in \mathbb{C}$ s.t. $\Lambda_\tau = c(\mathbb{Z} + \beta\mathbb{Z})$.

$\implies E_\tau \cong E_\beta.$ $\square$.

Denote $\overline{\Gamma_z} = \Gamma \cdot \{\pm 1\} / \{\pm 1\}$.

Then

Prop

For $z \in \mathfrak{h}$, we have 3 cases.

(1) $z \in \Gamma(1) i$, then $\overline{\Gamma(1)_z} \cong \mathbb{Z}/2\mathbb{Z}$.

(2) $z \in \Gamma(1) \rho$. then $\overline{\Gamma(1)_z} \cong \mathbb{Z}/3\mathbb{Z}$.

(3) $z \notin \Gamma(1) i$ and $z \notin \Gamma(1)\rho$, then $\overline{\Gamma(1)_z}$ is trivial.

For $z = \{\infty\}$, $\overline{\Gamma(1)_z} \cong \mathbb{Z}$.

Prop

For $z \in \mathfrak{h}^*$. $\Gamma \subseteq \Gamma' \subseteq \Gamma(1)$ finite index,

$f: X_\Gamma \longrightarrow X_{\Gamma'}$, Let p be the image of z in X_Γ.

Then the ramification index at p is $[\overline{\Gamma_z'} : \overline{\Gamma_z}]$

Prop

Let $d = [\Gamma(1) : \Gamma]$

$\nu_2 = \# \left(\{ z \in \mathfrak{h} \mid |\overline{\Gamma_z}| = 2 \} / \Gamma \right)$

$\nu_3 = \# \left(\{ z \in \mathfrak{h} \mid |\overline{\Gamma_z}| = 3 \} / \Gamma \right)$

$\nu_\infty = \#$ of cusp $\left(\mathbb{P}^1(\mathbb{Q}) / \Gamma \right)$

Then
$$g(X_\Gamma) = 1 + \frac{d}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.$$

§2 Over $\mathbb{Q}$

Define functors: For scheme S with N invertible.

$$\mathcal{F}_{\Gamma(1)} : S \longmapsto \{ E \mid E : \text{elliptic curve} / S \} / \text{isom}$$

$$\mathcal{F}_{\Gamma(N)} : S \longmapsto \{ (E/S, P, Q) \mid P, Q \text{ form basis of } E[N] \} / \text{isom}$$

$$\mathcal{F}_{\Gamma_1(N)} : S \longmapsto \{ (E/S, P) \mid P \text{ is of order } N \} / \text{isom}$$

$$\mathcal{F}_{\Gamma_0(N)} : S \longmapsto \{ (E/S, G) \mid \underbrace{G \le E}_{\Gamma\text{-structure on } E.} \text{ cyclic order } N \} / \text{isom}$$

Are they sheaves?

representable?

- $\mathcal{F}_{\Gamma(1)}$ is not a sheaf:

$\exists\ E_1 \not\cong E_2$ over k, but $E_1 \cong E_2$ over k^s.

Such phenomenon is parametrized by $H^1(\Gamma_k, \underbrace{\mathrm{Aut}(E_{k^s})}_{\cong\{\pm 1\}\ \text{assume}}) \cong k^*/(k^*)^2$.

Then for $d \in k^\times/(k^\times)^2$, (char $k = 0$)

$$y^2 = f(x) \cong dy^2 = f(x) \quad \text{over } k(\sqrt{d}).$$

not over k.

Rigidity

Prop

Any automorphism of E fixing

(1) $\Gamma(N)$-structure $N \geq 3$

(2) $\Gamma_1(N)$-structure $N \geq 4$

is identity.

pf

For $f: E \to E$, denote f^t the dual isogeny.

Then

$$[\deg f] = f f^t = f^t f.$$
$$[\mathrm{tr} f] = f + f^t.$$

We have

- $(\mathrm{tr} f)^2 \leq 4 \deg f.$

(1) Say $\varepsilon: E \to E$ fixes $\Gamma(N)$-structure, then

$\varepsilon - 1$ kills $E[N] \Rightarrow \varepsilon - 1 = g[N]$ for some g.

$$\begin{aligned} \deg g \cdot N^2 = \deg(\varepsilon - 1) &= \varepsilon^t \varepsilon + 1 - \mathrm{tr}(\varepsilon) \\ &= 2 - \mathrm{tr}(\varepsilon) \leq 4. \end{aligned}$$

$N \geq 3 \Rightarrow \deg g = 0$, $\varepsilon = 1$.

(2) $\varepsilon - 1$ has degree $\equiv 0 \mod N$. Idea is similar. $\square$.

Representability

Fact:

$R = \operatorname{Spec} \mathbb{Z}[\frac{1}{3}, B, C][1/\Delta]/(B^3 = (B+C)^3)$

represents $\mathcal{F}_{\Gamma(3)}$, where

$$E: \quad y^2 + a_1 xy + a_3 y = x^3.$$

with
$$\begin{cases} P = (0,0) \\ Q = (C, B+C) \\ a_1 = 3C - 1 \\ a_3 = -3C^2 - B - 3BC. \end{cases}$$

proof: play around with Riemann-Roch.

From the representability of $\Gamma(3)$, we prove that of $\Gamma(N)$ $N \geq 3$.

Relatively representable

Prop

Let E/S be an elliptic curve. (N invertible on S.) Let $\mathcal{F}$ be a functor on (Sch/S) defined by

$$T \longmapsto \begin{cases} (1) & \Gamma(N)\text{-structure on } E \times_S T. \\ (2) & \Gamma_1(N) \\ (3) & \Gamma_0(N) \end{cases}$$

Then F is representable by a finite étale scheme $/S$.

<u>pf</u>

(1) $T_0 = E[N] \times_S E[N]$ represents pair of N-torsion pts.

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "\text{finite étale}"'] \arrow[dr, phantom, "\square"] & (\mu_N^{\text{prim}})_S \arrow[d, hook] \\
T_0 \arrow[r, "\text{Weil pairing}"'] \arrow[d, "\text{finite étale}"'] & (\mu_N)_S \\
S &
\end{tikzcd}$$

(2) : quotient T by $H = \left\{ \begin{pmatrix} * & * \\ & 1 \end{pmatrix} \right\}$.

(3) $H = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\}$ $\square$

Thm

Suppose $(N,3)=1$, $N \geq 4$. Then $\mathcal{F}_{\Gamma(N)}$ is represented by a smooth affine scheme over $\mathbb{Z}[\frac{1}{3N}]$.

<u>pf</u>

<u>Step 1</u> $\mathcal{F}_{\Gamma(3N)}$ is representable.

Let $(E_{Y(3)}, (P, Q))$ corresponds to $id_{Y(3)} \in Hom(Y(3), Y(3))$.

Let T be the scheme in previous lemma.

Then

$$Hom(S,T) \cong \{(P_N, Q_N) : \Gamma(N)\text{-structure on } E \times_{Y(3)} S\}$$

$$+ \quad S \to T \to Y(3) \rightsquigarrow (P_3, Q_3) : \Gamma(3)\text{-structure on } E \times_{Y(3)} S$$

$\Rightarrow$ T represents $\mathcal{F}_{P(3n)}$ by CRT

Step 2 The action $GL_2(\mathbb{Z}/3\mathbb{Z})$ on $\mathcal{F}_{\Gamma(3n)}$ is free.

$$E[3N] = E[N] \times E[3]$$
$$\circlearrowleft$$
$$GL_2(\mathbb{Z}/3\mathbb{Z}) \ni g.$$

If $(E, (P_N, Q_N), (P_3, Q_3)) \xrightarrow[\sim]{\varepsilon} (E, (P_N, Q_N), g(P_3, Q_3))$.

rigidity of $(E, (P_N, Q_N))$ shows $\varepsilon = 1 \Rightarrow g = 1$.

Step 3.

$\mathcal{F}_{P(N)}$ is the quotient of $Y(3N)$ by $GL_2(\mathbb{Z}/3\mathbb{Z})$.

$\square$

Thm

For $N \geq 3$, $\mathcal{F}_{P(N)}$ is representable over $\mathbb{Z}[1/N]$ by a smooth affine scheme

sketch pf:

$\frac{dx}{y}$ invariant differential.

We can consider $\mathcal{F}_{\Gamma(2)}$ + sth.

$\rightsquigarrow$ If $(N, 6) = 1$ then it is representable over $\mathbb{Z}[\frac{1}{3N}]$ and $\mathbb{Z}[\frac{1}{2N}]$

$\rightsquigarrow$ over $\mathbb{Z}[\frac{1}{N}]$ - - - $\square$

Stack

A stack is a rule that sends morphism to a groupoid (Category with all morphisms iso).

s.t. $U_1 \longrightarrow U_2 \longrightarrow U_3$,

$$\mathrm{Res}^{U_2}_{U_1} \circ \mathrm{Res}^{U_3}_{U_2} \underset{\text{(not equal)}}{\simeq} \mathrm{Res}^{U_3}_{U_1}$$

with gluing property .

A sheaf of set $\mathcal{F}$ is a stack with

$\mathrm{Obj}\left(\mathcal{F}(U)\right) = \mathcal{F}(U)$, morphisms are identity .

<u>Example</u>

$X \circlearrowleft G$, G finite.

Consider category $[X/G]$: $\mathrm{Obj}([X/G]) = X$.

$\mathrm{Hom}(x, y) = \{ g \in G \mid gx = y \}$.

For Y another set,

$$\begin{tikzcd} Y \times_{[X/G]} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "b"] \\ Y \arrow[r, "a"'] & {[X/G]} \end{tikzcd}$$

$\mathrm{Obj}(Y \times_{[X/G]} X)$

$= \{ (y, x, \alpha) \mid \alpha \in \mathrm{Hom}(a(y), b(x)) \}$.

morphisms being identity .

$Y \times_{[X/G]} X$ has G-action: $g(y, x, \alpha) = (y, gx, g\alpha)$.

Then • every fiber of $y \in Y$ has simply-transitive G-action.

• $Y \times_{[X/G]} X \longrightarrow X$ is G-equivariant.

For scheme, we define $(Y \longrightarrow [X/G])$ is the category

Object:
$$\begin{tikzcd} T \arrow[r, "G\text{-equivariant}"] \arrow[d, "G\text{-torsor}"'] & X \\ Y & \end{tikzcd}$$

morphism:
$$\begin{tikzcd} T' \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & T \arrow[r] \arrow[d] \arrow[ur, phantom, "2"] \arrow[dl, phantom, "2"] & X \\ & Y & \end{tikzcd}$$

restriction: $Y' \longrightarrow Y \rightsquigarrow$ pullback.

Definition

A Deligne mumford stack X is a stack

s.t. $\exists\ \tilde{X} \longrightarrow X$ with $\tilde{X}$ a scheme,

s.t.
$$\begin{tikzcd} Y \times_X \tilde{X} \arrow[r] \arrow[d, "\text{surj étale}"'] \arrow[dr, phantom, "\square"] & \tilde{X} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$
, $Y \times_X \tilde{X}$ is a scheme.

$\mathcal{M}_{\Gamma(1)}$: $S \longmapsto$ groupoid of elliptic curve $/S$.

is a Deligne-Mumford stack with

$$\begin{tikzcd} \text{the one in lemma} & T \arrow[l, hook'] \arrow[r] \arrow[d, "\text{finite}"', "\text{étale}"] & Y(N) \arrow[d] \\ & S \arrow[r] & \mathcal{M}_{\Gamma(1)}. \end{tikzcd}$$

$$\updownarrow$$

$$E/S$$

($\Gamma_1(N)$ similar).

$\mathcal{M}_0(N)$: $S \longmapsto$ groupoid of elliptic curve with $\Gamma_0(N)$.

It is a Deligne-Mumford stack: for $p \nmid N$,

Let $Y(S) \longmapsto \{(E, G, (P,Q)) \mid (P,Q): \Gamma(p)\text{-structure}, G: \Gamma_0(N)\text{-str}\}/\sim$.

It is represented over $\mathbb{Z}[\frac{1}{pN}]$ by

$$Y = Y(pN) \Big/ \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\} \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \qquad \text{(affine)}.$$

Then

$$\begin{tikzcd} T \arrow[r] \arrow[d, "\text{finite}"', "\text{étale}"] & Y \arrow[d] \\ S \arrow[r] & \mathcal{M}_0(N) \end{tikzcd}$$

$$\updownarrow$$

$$(E/S, G).$$

T represents $\Gamma(N)$-structure on (Ell/S)

$Y_0(N) \triangleq Y/GL_2(\mathbb{Z}/p\mathbb{Z})$ is affine smooth $/\mathbb{Z}[\frac{1}{pN}]$.

patching for various $p \rightsquigarrow$ over $\mathbb{Z}[\frac{1}{N}]$.

Y_0 represents the sheafification of presheaf

$$S \longmapsto \{(E/_S, G)\}/\sim .$$

is called the coarse space.

Compactification

valuative criterion $\rightsquigarrow$ extend elliptic curve over DVR.
For stack, we somehow allow field extension.

Recall semi-stable reduction thm:
after a finite extension, E has $\underbrace{\text{good}}_{\text{elliptic curve}}$ or $\underbrace{\text{multiplicative reduction}}_{\text{nodal cubic}}$.

Define n-gon $C_n\ /k$: $(\mathbb{P}^1_k) \times \mathbb{Z}/n\mathbb{Z}. \big/ _{(\infty, i) \sim (0, i+1)}$

$(C_1 : \text{nodal cubic})$

$C_n^{sm} = \mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$ a gp

$$0 \longrightarrow \mu_n \longrightarrow C_n^{sm}[n] \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 .$$

Define generalised elliptic curve $/S$: $(E, +, e)$

E/S is proper flat

$e \in E(S)$.

$+ : \quad E^{sm} \times E \longrightarrow E$

s.t. • $(+, e)$ gives gp structure on E^{sm}.

• The geometric fiber is elliptic curve or n-gon

• The generic fiber is elliptic curve

Then define

$\overline{\mathcal{M}(1)}(S)$: groupoid of generalized elliptic curve$/S$.

Then

Thm

$\overline{\mathcal{M}(1)}$ is a proper D-M. stack $/\mathbb{Z}$.

Higher level

Consider $\Gamma_0(N) = \{ g \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \}$. N prime.

$\mathbb{P}^1(\mathbb{Q}) / \Gamma_0(N) = \infty \Gamma_0(N) \cup 0 \Gamma_0(N)$

$\Rightarrow$ 2 cusps.

1-gon has only 1 cyclic gp of order N
μ_N.

So we consider N-gon, which has 2 $\Gamma_0(N)$-structure

μ_N, $\mathbb{Z}/N\mathbb{Z}$.

However, we shall think of them as

$(C_N, \mathbb{Z}/N\mathbb{Z})$ & (C_1, μ_N)
∞ $\qquad$ 0

s.t. the level structure meets all irred comp.

Define

$\overline{\mathcal{M}_0(N)}$ = groupoid of generalized elliptic curve with $\Gamma_0(N)$-structure that meets all irred comp.

Then $\overline{\mathcal{M}_0(N)}$ is a proper D-M stack.

Maps

Over $\mathbb{C}$, $N'|N \Rightarrow X_0(N) \longrightarrow X_0(N')$.

We describe this map for generalized elliptic curve. For elliptic curve, it is clear.

In general, $(\mathcal{E}, G)$, $H \leq G$ unique subgp
$\uparrow$ order N of order N'.

Then we contract those component that do not meet H.

$\left(\begin{array}{l} f: \mathcal{E} \to S. \quad \mathcal{A} = \text{sheaf of graded ring } \bigoplus_{n \geq 0}^{\infty} f_*(\mathcal{O}_{\mathcal{E}}(nH)). \\ \text{Then the contraction is } \mathrm{Proj}(\mathcal{A}) \end{array} \right)$

§3 Over $\mathbb{Z}$.

In comparison to $/ \mathbb{Z}[\frac{1}{N}]$, we want to look at elliptic curve / field with characteristic N.

$$E[N](\bar{k}) < N^2.$$

Assume N square free

If we replace $G \leq \mathcal{E}$ closed étale of order N
$\downarrow$
closed flat.

Then $\overline{\mathcal{M}}_0(N)$ is flat D-M stack $/\mathbb{Z}$.

Fiber in bad characteristic

Over $\mathbb{F}_p$, we have Frobenius and Verschiebung

$$\begin{tikzcd} E \arrow[r, "F_p"] \arrow[rr, bend right, "{[p]}"'] & E^{(p)} \arrow[r, "V_p"] & E \end{tikzcd}.$$

Recall

E is ordinary if $E[p](\bar{k}) \cong \mathbb{Z}/p\mathbb{Z}$.

Then

$$E[p] \cong \underset{\substack{\text{local} \\ \ker F_p}}{\mu_p} \times \underset{\substack{\text{étale} \\ \ker V_p \ (\text{if } E = E^{(p)})}}{\mathbb{Z}/p\mathbb{Z}}$$

There are exactly 2 closed subgp of order p.

I.e. μ_p, $\mathbb{Z}/p\mathbb{Z}$.

The same holds for p-gon.

E is supersingular if $E[p](\bar{k}) = 0$

Then

$$0 \longrightarrow \alpha_p \longrightarrow E[p] \longrightarrow \alpha_p \longrightarrow 0$$

non-split. So there is only 1 closed subgp of order p, $\alpha_p = \ker(F_p)$

Assume N squarefree, and $N = pN'$.

Then given $\Gamma_0(N')$-structure (E, G) with E not supersingular, there are 2 choices of $\Gamma_0(N)$-structure i.e.

$$\begin{array}{ccc} & (E,\ G\cdot \mathbb{Z}/p\mathbb{Z}), & (E,\ G\cdot \mu_p). \\ (E = E'^{(p)}) & \| & \| \\ & (E'^{(p)},\ V_p^{-1}(G)) & (E,\ G,\ \ker F_p). \end{array}$$

For E supersingular, these 2 choices coincide.

Thus we define

$$f, g : \overline{\mathcal{M}_0(N')_{\mathbb{F}_p}} \longrightarrow \overline{\mathcal{M}_0(N)_{\mathbb{F}_p}}.$$

$$(E, G) \xmapsto{\ f\ } (E, G, \ker F_p)$$

$$(E, G) \xmapsto{\ g\ } (E^{(p)}, V^{-1}(G)).$$

$$f', g' : \overline{\mathcal{M}_0(N)_{\mathbb{F}_p}} \longrightarrow \overline{\mathcal{M}_0(N')_{\mathbb{F}_p}}.$$

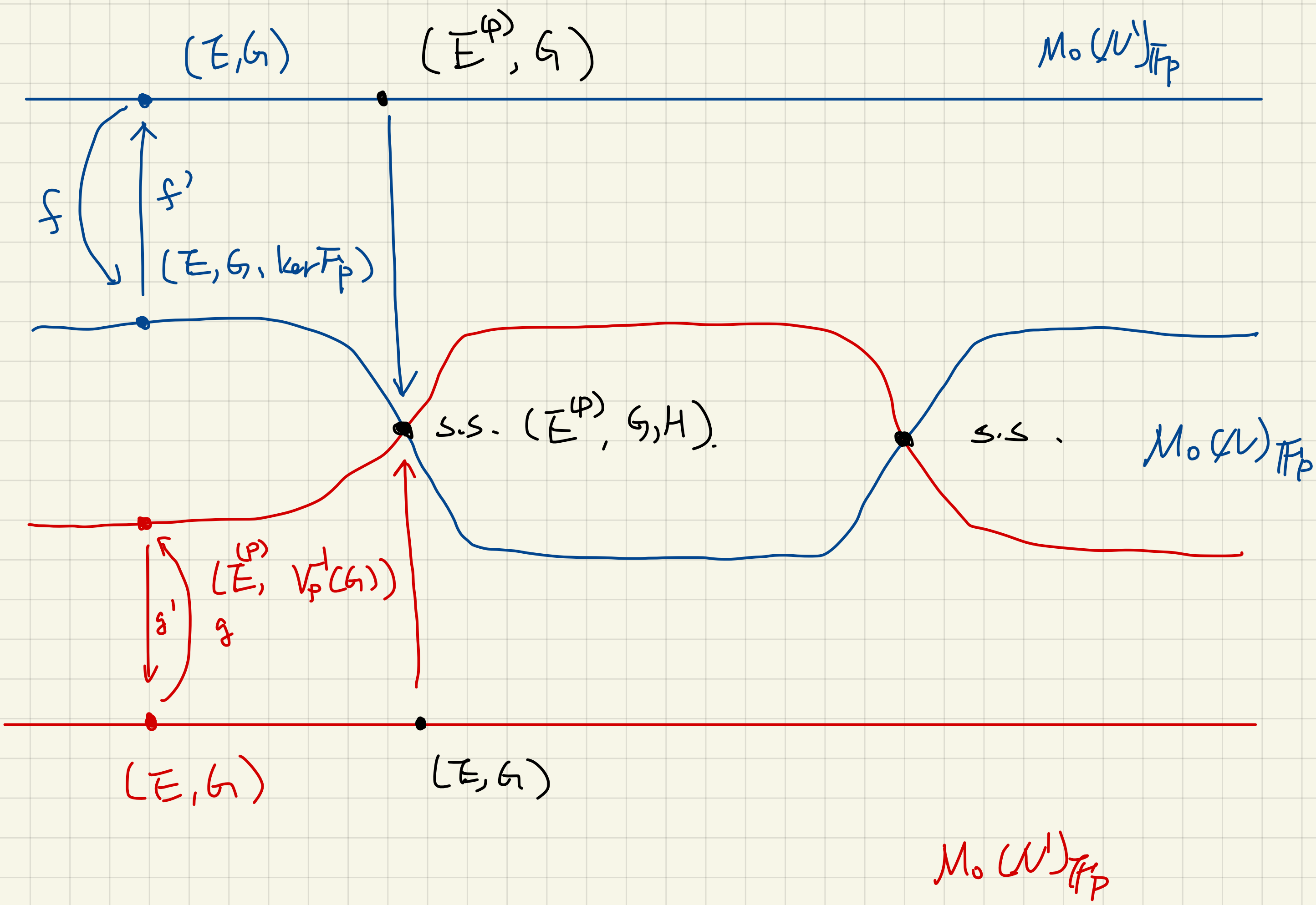
$$(E, G, H) \xmapsto{\ f'\ } (E, G).$$

$$(E, G, H) \xmapsto{\ g'\ } (E/H, \text{image of } G \text{ in } E/H).$$

Then $f' \circ f = id$. $\quad g' \circ g = id$ $\left(V_p : E^{(p)}/\ker V_p \xrightarrow{\sim} E \right)$.

$$g' \circ f = T_p = f' \circ g$$

In picture,



(E,G)
$(E^{(p)}, G)$
$M_0(\mathcal{U}')_{\mathbb{F}_p}$
f
f'
$(E, G, \ker F_p)$
s.s. $(E^{(p)}, G, H)$.
s.s.
$M_0(\mathcal{U})_{\mathbb{F}_p}$
$(E^{(p)}, V_p^{-1}(G))$
g'
g
(E,G)
(E,G)
$M_0(\mathcal{U}')_{\mathbb{F}_p}$